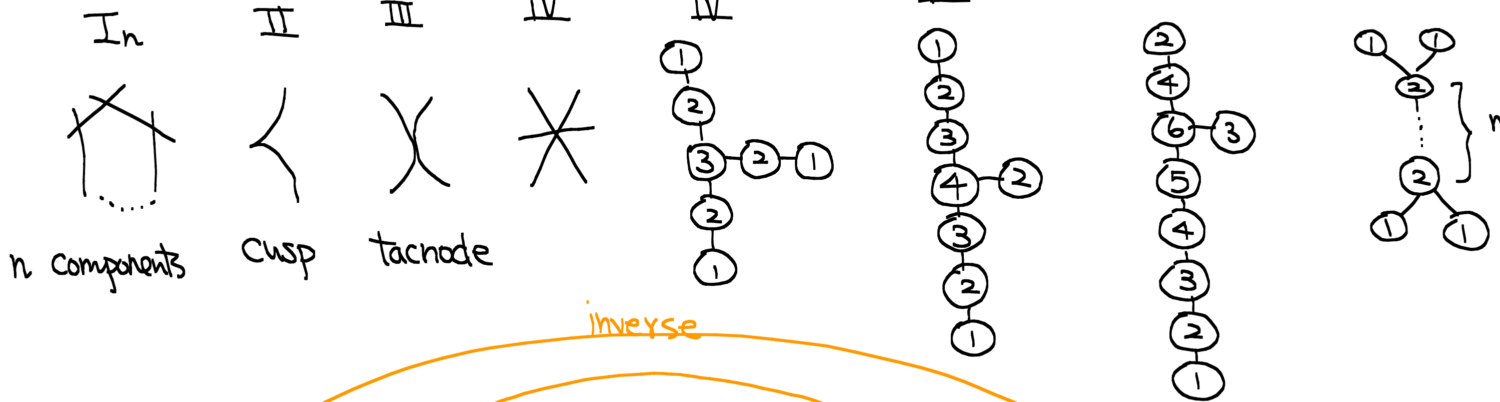
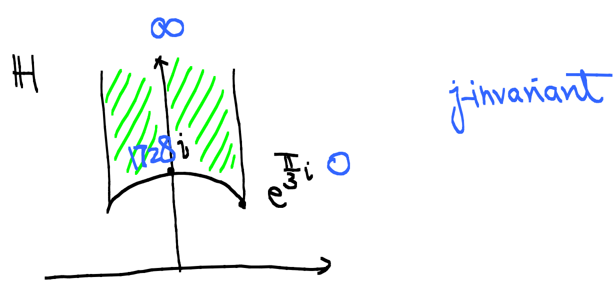
Lecture 12: More about Elliptic Surfaces

The possible singular fibres in a minimal elliptic fibration is classified by Kodaira.



inverse

	I_n	II	III	IV	IV^*	III^*	II^*	I_n^*
	$\begin{pmatrix}1&n\\0&1\end{pmatrix}^{\infty}$	$\begin{pmatrix}1&1\\-1&0\end{pmatrix}^{6}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}^{4}$	$\begin{pmatrix}0&1\\-1&-1\end{pmatrix}^{3}$	$\begin{pmatrix}-1&-1\\1&0\end{pmatrix}^{3}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}^{4}$	$\begin{pmatrix}0&-1\\1&1\end{pmatrix}^{6}$	$\begin{pmatrix}-1&-\nu\\0&-1\end{pmatrix}^{\infty}$
group structure	$\mathbb{Z}_n\times\mathbb{C}^*$	$\mathbb{C}$	$\mathbb{Z}_2\times\mathbb{C}$	$\mathbb{Z}_3\times\mathbb{C}$	$\mathbb{Z}_3\times\mathbb{C}$	$\mathbb{Z}_2\times\mathbb{C}$	$\mathbb{C}$	$(\mathbb{Z}_2)^2\times\mathbb{C}$, n even; $\mathbb{Z}_4\times\mathbb{C}$, n odd
j-inv.	∞	0	1728	0	0	1728	0	∞



Realization of the singular fibres

$$\mathbb{C}/_{\mathbb{Z}+\mathbb{Z}\tau} \cong \left\{ y^2 = 4x^3 - g_2(\tau)\,x - g_3(\tau) \right\} =: E$$

$\tau \in \mathbb{H}$

$$g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-4}$$

$$g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-6}$$

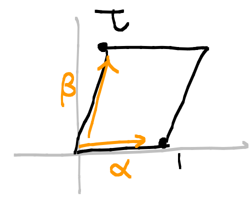
$$\Delta(\tau) = g_2(\tau)^3 - 27\, g_3(\tau)^2 = 0$$

iff E is singular

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)} = q^{-1} + 744 + 196884q + \cdots$$

$q = e^{2\pi i \tau}$

or $\tau = \frac{1}{2\pi i}\log q$



$\langle \alpha, \beta \rangle = 1$

Type I_1

$$X \supseteq X_0 = T^*\Delta / \mathbb{Z} + \mathbb{Z}\tau(z) \qquad \tau(z) = \frac{1}{2\pi i}\log z$$

Compactification over $z=0$

$$\downarrow$$

Δ°_z = punctured disc

then $X \setminus X_0$ is an I_1-fibre.

monodromy on H_1(fibre) is $T: \begin{array}{l} \alpha \longmapsto \alpha \\ \beta \longmapsto \alpha+\beta \end{array}$ Picard-Lefschetz formula

α: vanishing cycle as $z \to 0$

$$X = \{ yz^2 = 4x^3 + (t-3)xz^2 + (t-1)z^3 \} \subseteq \mathbb{P}^2_{(x,y,z)} \times \Delta_t$$

smooth elliptic surface

fibre at 0 is nodal w/ singularity at $(1, -4, 0)$

Type I_k

$$\begin{array}{ccccc} X_k \longrightarrow & X'_k & \longrightarrow & X \supseteq I_1 & \\ & \downarrow & & \downarrow & \\ & \Delta_{t'} & \longrightarrow & \Delta_t \ni & 0 \\ & t' & \longmapsto & t'^k = t & \end{array}$$

minimal resolution

locally near the node of the I_1 fibre, X is modeled on $\{xy = t\}$. Therefore, X'_k has a singularity locally modeled by $xy = t'^k$

In general, singular fibre of type I_k happens when k singular fibres of type I_1 w/ parallel vanishing cycles.

Thus, the monodromy $\sim \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

Normalize a germ of I_k fibre as

$$\begin{array}{ccl} X & \supseteq & X^* = \mathbb{C} \times \Delta^* / \mathbb{Z} + \mathbb{Z} \cdot \frac{k}{2\pi i} \log z \\ \pi \downarrow & & \downarrow \\ \Delta_z & \supseteq & \Delta^* \end{array}$$

then we can identify a section of X^* w/ a possibly multi-valued function $f: \Delta^* \longrightarrow \mathbb{C}$

Lemma 2: (i) $\sigma: \Delta^* \longrightarrow X^*$ holo. section over Δ^*

then $\sigma(z) = h(z) + \frac{a}{2\pi i} \log z + \frac{b}{(2\pi i)^2} (\log z)^2$.

w/ $h(z)$ holo. function on Δ^*, $\frac{2b}{k} \in \mathbb{Z}$, $a+b \in \mathbb{Z}$

(ii) σ extends to a holo. section of X

iff • $h(z)$ extends to a holo. function on Δ

• $b = 0$

Q: What is the analogue for other singular fibres?

pf: (i) f = section over Δ^*, then analytic continuation gives

$$f(e^{2\pi i} z) = f(z) + n + \frac{mk}{2\pi i} \log z \quad , \quad m, n \in \mathbb{Z}$$

Notice that

$$\frac{a}{2\pi i}\log z + \frac{b}{(2\pi i)^2}(\log z)^2$$

$\mathbb{Z} + \mathbb{Z}\frac{k}{2\pi i}\log z$

$\Downarrow$

$$\xrightarrow{\text{analytic continuation}} \frac{a}{2\pi i}\log z + \frac{b}{(2\pi i)^2}(\log z)^2 + a+b + \frac{2b}{2\pi i}\log z$$

is a section iff $a+b \in \mathbb{Z},\ \frac{2b}{k} \in \mathbb{Z}$

If $n, m \in \mathbb{Z}$, $\exists!\ a, b$ s.t $\begin{array}{l} 2b = mk \\ a+b = n \end{array}$

then $h(z) = f(z) - \frac{a}{2\pi i}\log z - \frac{b}{(2\pi i)^2}(\log z)^2$

is single-valued on Δ^*.

(ii) We need coordinates near the nodes of the I_k-fibre.

$$U_i = \left\{ (u_i, v_i) \in \mathbb{C}^2 \,\middle|\, |u_i| < 1, |v_i| < 1 \right\}$$

$X := \coprod_i U_i / \sim$

① $\begin{array}{c} U_i \\ \Psi \\ (u_i, v_i) \\ \neq \\ 0 \end{array} \sim \begin{array}{c} U_{i+1} \\ \Psi \\ (u_{i+1}, v_{i+1}) \end{array}$ if $\begin{array}{l} u_{i+1} = v_i^{-1} \\ v_{i+1} = u_i v_i^2 \end{array}$

each component is a (-2)-curve

② $\begin{array}{c} U_0 \\ \Psi \\ (u_0, v_0) \end{array} \sim \begin{array}{c} U_k \\ \Psi \\ (u_k, v_k) \end{array}$

Check: X is smooth.

$$\begin{array}{ccc} X & & (u_i, v_i) \in U_i \\ \downarrow & & \downarrow \\ \Delta & \ni & z = u_i v_i \end{array}$$

$u_0 = e^{2\pi i x}$, $v_0 = z e^{-2\pi i x}$

$$\frac{du_i \wedge dv_i}{u_i v_i} = \frac{dx \wedge dz}{z}$$

Now from (i), a section $\sigma(z)$ on U_0 is of the form

$$u_0 = \exp\left[2\pi i\left(h(z) + \frac{a}{2\pi i}\log z + \frac{b}{(2\pi i)^2}(\log z)^2\right)\right] \quad \text{holo. on } \mathcal{U}_0$$

$$= \underbrace{e^{2\pi i h(z)}}_{\neq 0} \cdot z^a \qquad \text{if } b=0,\ h(z) \text{ holo at } z=0$$

$= g(z)$, $g(0) \neq 0$

$$v_0 = e^{-2\pi i h(z)} z^{1-a}$$

then $\sigma|_{\mathcal{U}_i} = \left(g\cdot z^{a-i},\ g^{-1} z^{1-a+i}\right)$

Therefore, $\sigma(0) \in E_a = \{u_a(0) \neq 0,\ v_a(0) = 0\}$.

<u>Type I_0^*</u>

$$\begin{array}{c} \overset{i}{\text{involution}} \circlearrowright X = \mathbb{C}\times\Delta/\Lambda \\ \downarrow \\ \Delta_u \end{array}, \quad \Lambda = \mathbb{Z} + \mathbb{Z}\tau(u)$$

$\tau(u) = \tau(0) + u^{2k}$, $k \in \mathbb{N}$

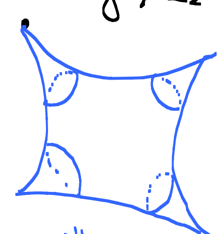
$i : (c,u) \longmapsto (-c,-u)$ fibre preserving

locally $\mathbb{C}/\mathbb{Z}_2$

 $\subseteq \tilde{X} \longrightarrow X/i$ fibre over 0 is $\underbrace{E_{\tau(0)}/\mathbb{Z}_2}_{4\ \mathbb{Z}_2\text{-orbifold points}} \cong \mathbb{P}^1$

minimal resolution

$X/i \downarrow \Delta$


pillowcase

Thus, the monodromy $\sim \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$

<u>Type $I_{n\geq 1}^*$</u>

$$\begin{array}{c} \sigma \circlearrowright X = \mathbb{C}\times\Delta/\Lambda \\ \downarrow \\ \Delta_u \end{array}, \quad \Lambda = \mathbb{Z} + \mathbb{Z}\tau(u)$$

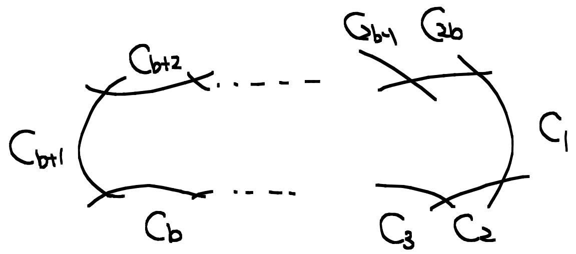
$\tau(u) = \frac{2n}{2\pi i}\log u$ i.e central fibre of type I_{2n}

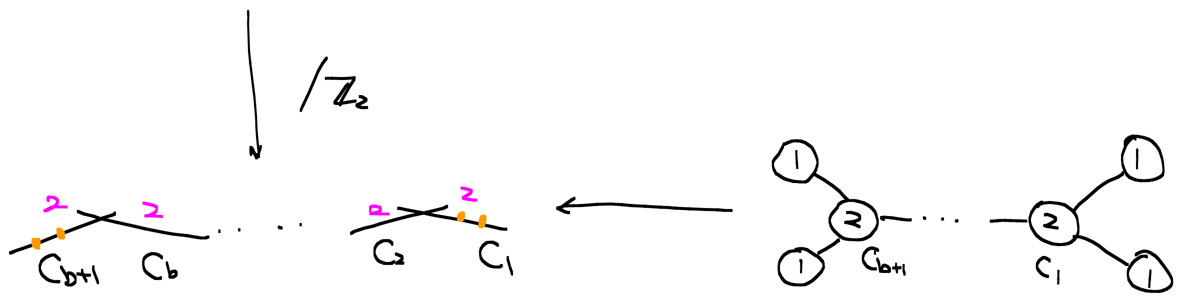
$\sigma : (c,u) \longmapsto (-c,-u)$ extends over the singular fibre

$\sigma(C_i) = C_{2n+2-i}$, $i = 2, \cdots, b$ $\sigma(C_1) = C_1$, $\sigma(C_{b+1}) = C_{b+1}$

acts like $z \longmapsto \frac{1}{z}$

has 2 fixed points





blow up the two

$\mathbb{Z}_2$-orbifold point on C_1, C_{b+1}

Finite monodromy

Type II, III, IV, IV*, III*, II*

ex. Type II

$X = \Delta \times \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau(u)$,

$\mu \circlearrowright X \to \Delta \ni u$

$$\tau(u) = \frac{e(\frac{1}{3}) - e(\frac{2}{3}) f(u)}{1 - f(u)}, \quad f(u) = \sum_{n \equiv 2 \pmod 6} a_n u^n$$

$(e(\tfrac{1}{6})u, 1) \sim (u, -\tau(u))$

$(e(\tfrac{1}{6})u, \tau(e(\tfrac{1}{6})u)) \sim (u, 1+\tau(u))$

$\therefore$ monodromy $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

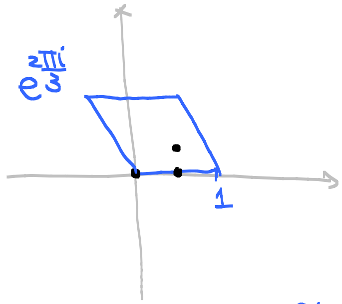
$$\mu : (u, x) \longmapsto \left(e(\tfrac{1}{6})u, -\frac{x}{\tau(u)}\right)$$

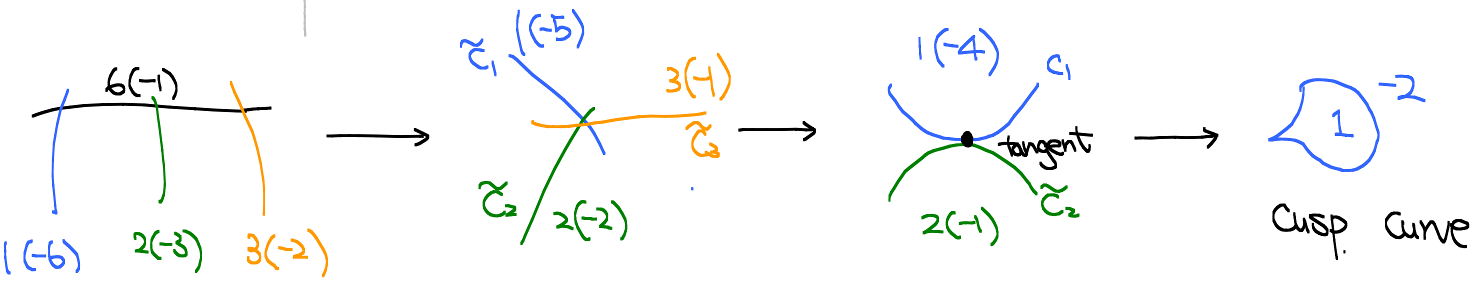
$$\tau(e(\tfrac{1}{6})u) = \frac{-1}{\tau(u)} - 1$$

Then central fibre of $X/\langle\mu\rangle$ is of multiplicity 6

$\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}e(\frac{1}{3})$

Fixed points on X_0 :





Cusp curve

multiplicity (self-intersection)

$$C_1 . C_2 = \pi^* C_1 . \pi^* C_2$$
$$= (\tilde{C}_1 + \tilde{C}_3).(\tilde{C}_2+\tilde{C}_3)$$
$$= \underset{+1}{\tilde{C}_1.\tilde{C}_2} + \underset{+1}{\tilde{C}_1.\tilde{C}_3} + \underset{-1}{\tilde{C}_3^2} + \underset{+1}{\tilde{C}_2.\tilde{C}_3} = 2$$

Logarithmic Transformation

Given $\begin{array}{c} X \\ \downarrow \\ \Delta \end{array}$ elliptic surface , X_0 smooth or of type I_k

Then $\exists$ $\begin{array}{c} X' \\ \downarrow \\ \Delta \end{array}$ elliptic surface s.t. $\begin{array}{ccc} X' \setminus X_0' & \cong & X \setminus X_0 \\ & \searrow \; \swarrow & \\ & \Delta_0 & \end{array}$, X_0' : multiple fibre mI_0 or mI_k

We will explain the case X_0 is smooth.

- Identify $\begin{array}{c} X \\ \downarrow \\ \Delta \end{array} \cong \Delta \times \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau(z)$, $\operatorname{Im}\tau(z) \neq 0$

- $X' = \left(\Delta \times \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau(z^m)\right)/\sim$ $\quad (z,x) \sim \left(e(\tfrac{1}{m})z,\ x + \tfrac{k}{m}\right)$

$(k,m) = 1$ $\quad$ no fixed points

then $\quad X' \setminus X_0' \cong X \setminus X_0$

$$(z,x) \longmapsto \left(z^m, x - \tfrac{k}{2\pi i}\log z\right)$$

before quotient

One can glue the local model to replace the central fibre by a multiple fibre.

- Conversely, any multiple fibre can be replaced by a smooth one via redoing the above procedure after suitable coordinate change.

- Logarithmic transformation is NOT a birational operation.

  the isomorphism $X' \setminus X_0 \cong X \setminus X_0$ does NOT extend as a rational map $X' \dashrightarrow X$

  It can change an algebraic surface to a non-algebraic one or even change the topology.

ex. $X = E \times \mathbb{P}^1$, $\kappa(X) = -\infty$

$Y =$ logarithmic transform on X many times.

Fact: $\begin{array}{c} X \\ f\downarrow \\ B \end{array}$ minimal elliptic fibration. $X_{b_i} = m_i F_i$ multiple fibre

Then $K_X = f^*\left(K_B \otimes (R^1 f_* \mathcal{O}_X)^\vee\right) \otimes \mathcal{O}_X\left(\sum_i (m_i - 1) F_i\right)$

Therefore, K_Y can be generated by global sections

& $\kappa(Y) = 1$!!

ex. (Halphen pencil)

$F, G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$

then $sF^n + tG = 0$ defines a pencil of <u>elliptic curves</u>.

can see via base change

Blow up the base points (each w/ multiplicity n)

gives a rational surface w/ elliptic fibration

and a fibre of multiplicity n.

In particular, it has no section due to the topological reason.

<u>Semi-Flat Metric</u>

Given an elliptic fibration w/o singular fibres w/ a section

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ B \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

possibly non-compact

The Abel–Jacob map identifies

$$\begin{tikzcd} X \arrow[r] \arrow[d] & T^*B/\Lambda = X_{mod} \arrow[d] \\ B \arrow[r] \arrow[u, bend left, "\sigma"] & B_u \arrow[u, bend right, "0"'] \end{tikzcd}$$

$$\Lambda = \mathbb{Z}\tau_1(u)\,du \oplus \mathbb{Z}\tau_2(u)\,du \qquad \mathrm{Im}(\bar{\tau}_1\tau_2) > 0$$

τ_1, τ_2 (multi-valued) holo. functions

u: coordinate on B

x: fibre coordinate

(Greene–Shapere–Vafa–Yau)

$(X_{mod}, \omega_{sf,\varepsilon}, \Omega)$ hyperkähler triple

$$\Omega = k(u)\, du \wedge dx$$

$$\omega_{sf,\varepsilon} = i|k|^2 \frac{\mathrm{Im}(\bar{\tau}_1 \tau_2)}{\varepsilon} du \wedge d\bar{u} + \frac{i}{2} \frac{\varepsilon}{\mathrm{Im}(\bar{\tau}_1\tau_2)} (dx - B du) \wedge (d\bar{x} - \bar{B} d\bar{u})$$

semi-flat metric fibrewise-flat

$$B(u,x) = \frac{1}{\mathrm{Im}(\bar{\tau}_1\tau_2)} \left(\mathrm{Im}(\bar{\tau}_1 x) \frac{\partial \tau_2}{\partial u} - \mathrm{Im}(\bar{\tau}_2 x) \frac{\partial \tau_1}{\partial u} \right)$$

Check: • $\omega_{sf,\varepsilon}$ is invariant under $x \mapsto x + m\tau_1(u) + n\tau_2(u)$, $m, n \in \mathbb{Z}$.

• $2\omega_{sf,\varepsilon}^2 = \Omega \wedge \bar{\Omega}$.

The semi-flat metric depends on the modulus of the elliptic fibration, size of the fibre relative to the base, and a choice of section.

τ_1, τ_2 ε

identify w/ the zero section.

Remark: There exists multi-section $\hbar$ on X_{mod} s.t

$T_\hbar^* \omega_{sf,\varepsilon}$ is still well-defined!

non-standard semi-flat metric